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UCRL-TR-200179

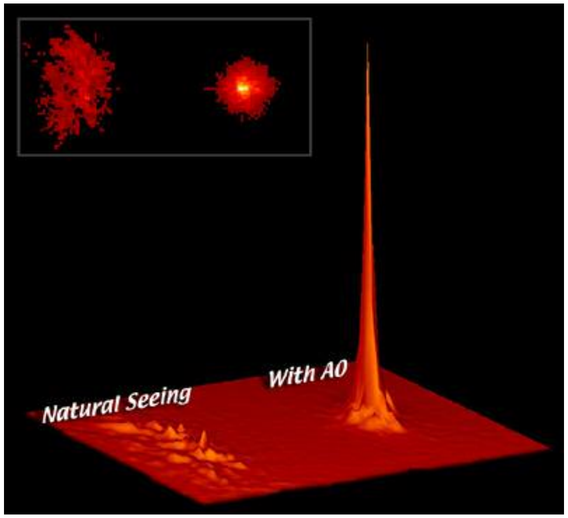
Huygens Integral Transformation for A 4x4 Ray Matrix

Don Phillion, Gary Sommargren

October 8, 2003

Disclaimer

This work was performed under the auspices of the U.S. Department of Energy by University of California, Lawrence Livermore National Laboratory under Contract W-7405-Eng-48.



Point Source Diffraction Interferometry with Lensless Imaging

Don Phillion and Gary Sommargren

Center for Adaptive Optics October 1, 2003 Workshop

Phase shifting diffraction interferometry (PSDI)

The fundamental process of diffraction is used to generate independent measurement and reference wavefronts

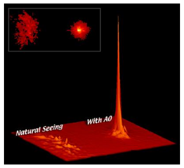
DIFFRACTION BY A λ-SIZE APERTURE CAN GENERATE A "PERFECT" SPHERICAL WAVEFRONT OVER A SPECIFIED NUMERICAL APERTURE

Diffractive configuration removes the limitations of conventional interferometry …

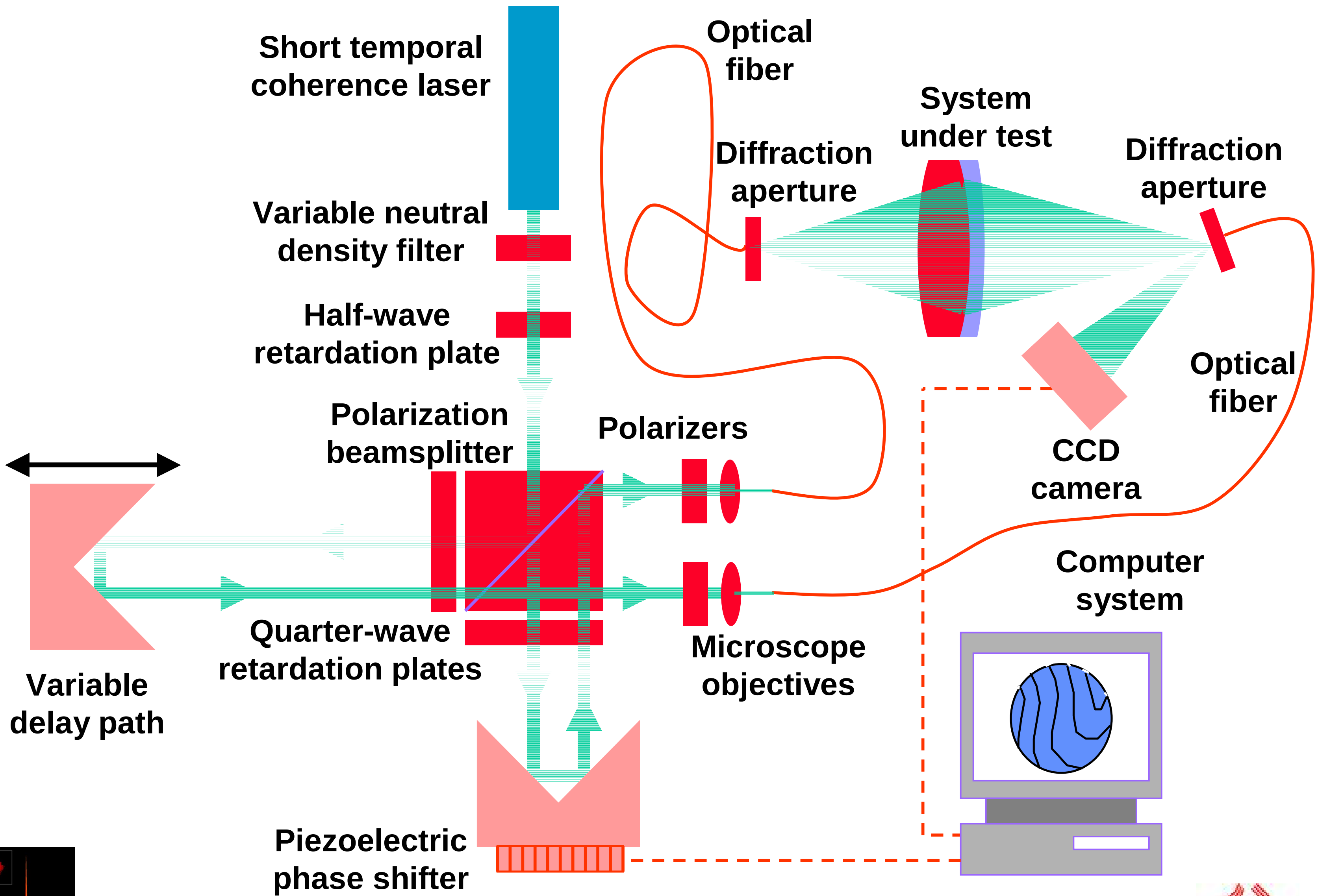
- **reference surface is eliminated**
- **auxiliary optics are kept to a minimum**

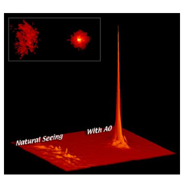
…while retaining all positive features including

- **phase shifting**
- **standard algorithms for data analysis**

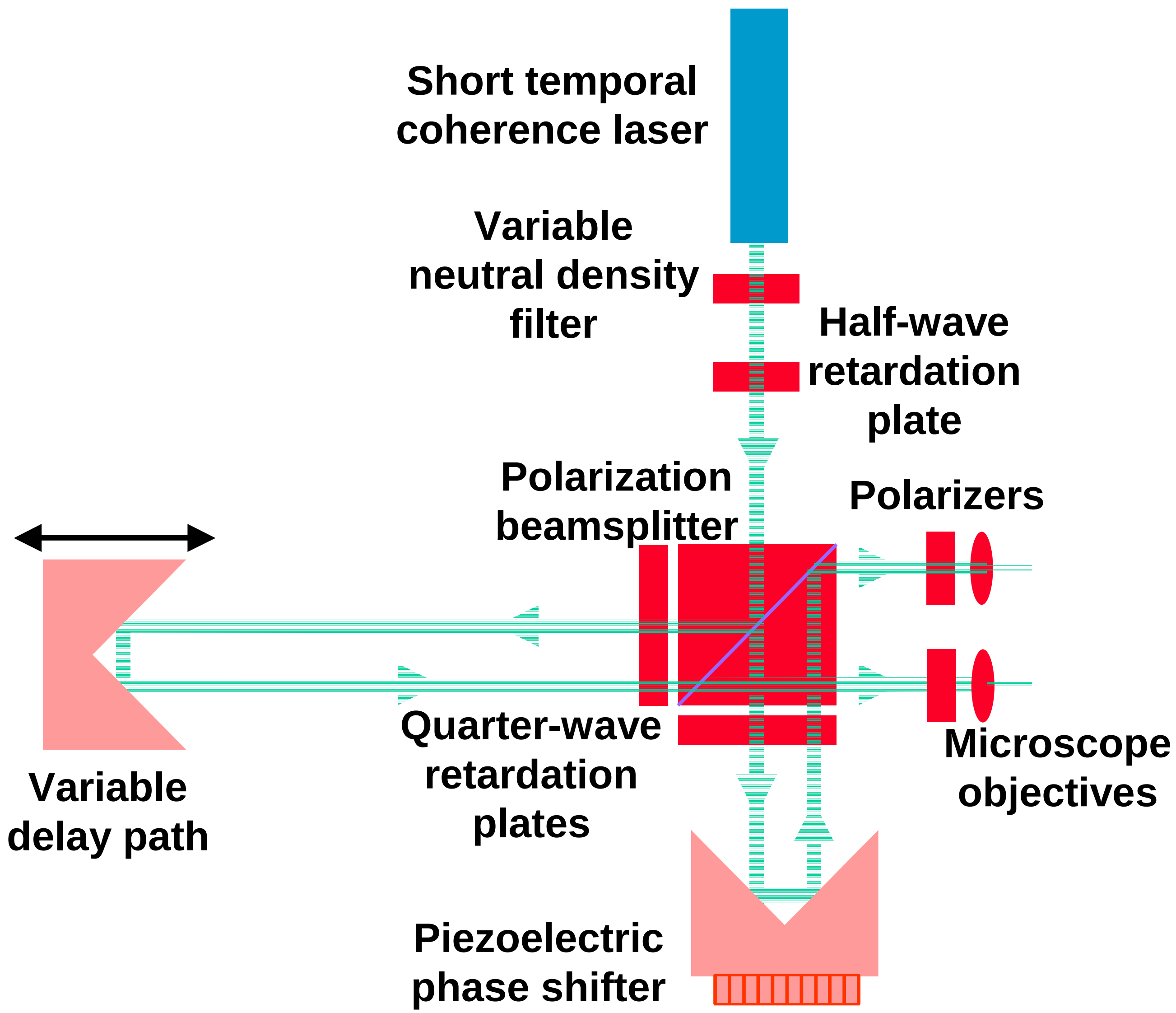


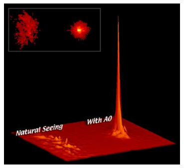
PSDI configured to measure optical systems





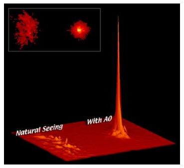
PSDI front end - laser and beam conditioning optics



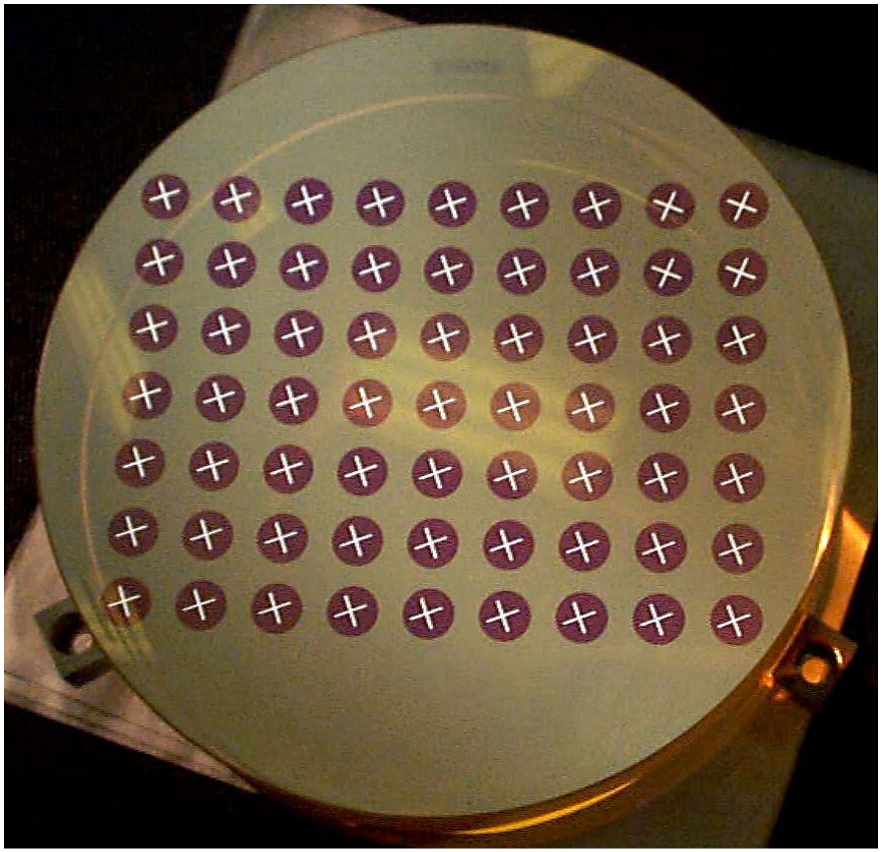


Intensity frames to unwrapped, propagated phase and amplitude at the pupil

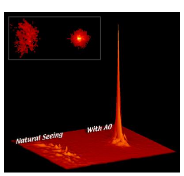
- Use the twelve bucket, $\pi/4$ phase step, least squares phase-shifting interferometry algorithm to get the wrapped phase and amplitude at the CCD.
- Usually the wrapped phase and amplitude data from at least eight sets of intensity frames are averaged together. A set of intensity frames is measured about once every two seconds. We implement what we call *delayed phase unwrapping* and do what we call *complex phasor averaging*.
- Propagate to the pupil using the ABCD matrix and the Huygens integral transformation implemented in two steps using the fast Fourier transform.
- Choose the set of pixels to be phase unwrapped
- Do the phase unwrapping



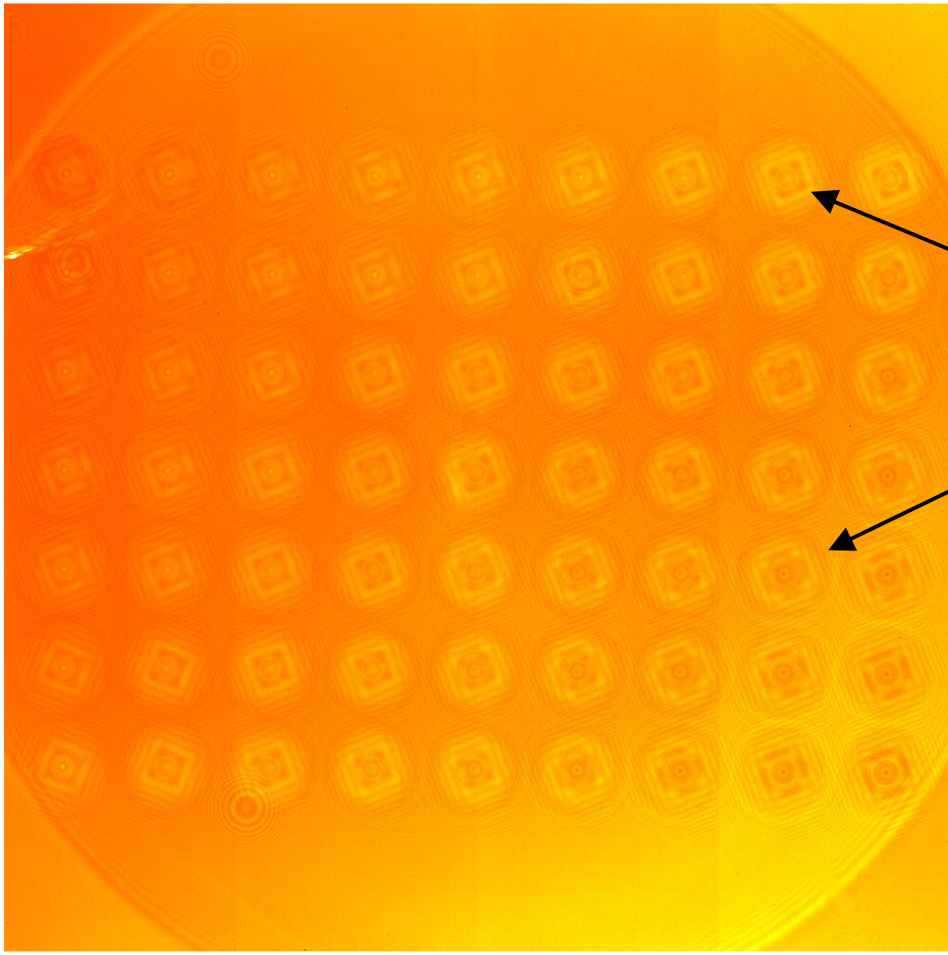
A distortion calibration grid (DCG) is used to test numerical back-propagation for the lensless PSDI



The DCG is an array of aluminum fiducials on a spherical mirror that matches the radius of curvature of the aspheric mirror under test



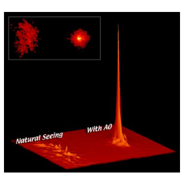
Interferogram of the DCG acquired by the CCD camera



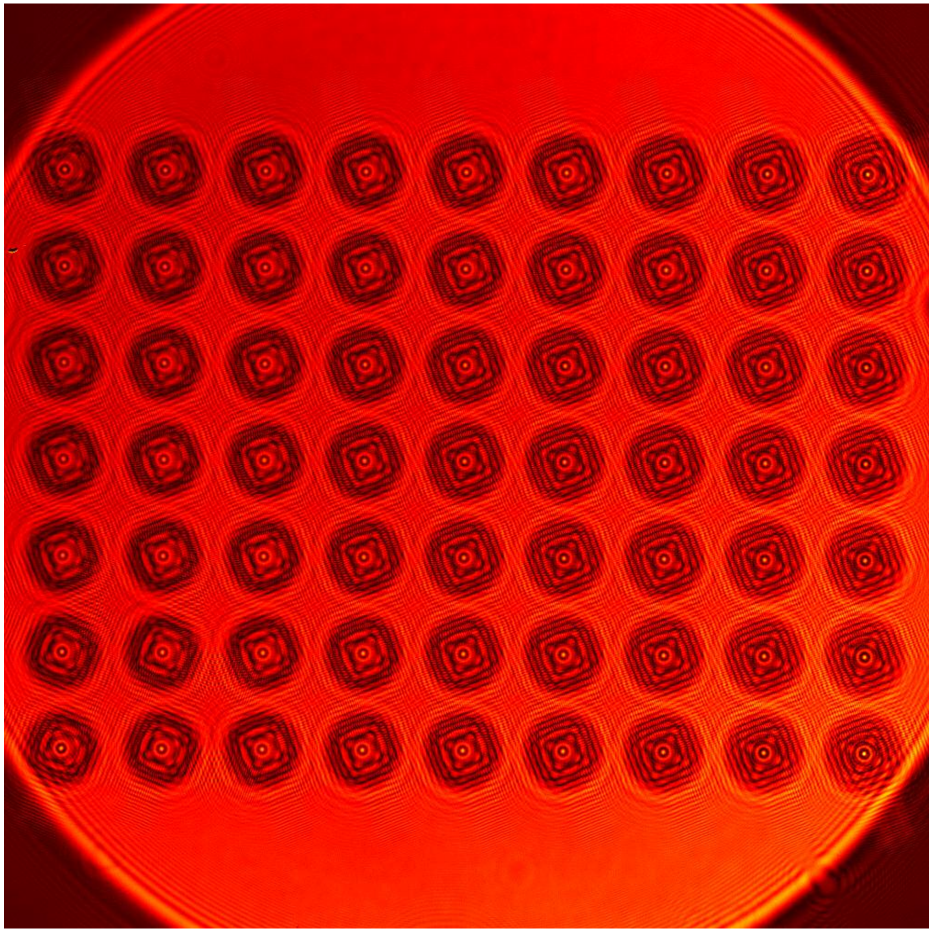
Diffraction patterns of fiducials

12 phase shifted interferograms are acquired to calculate the amplitude and phase of the field at the CCD

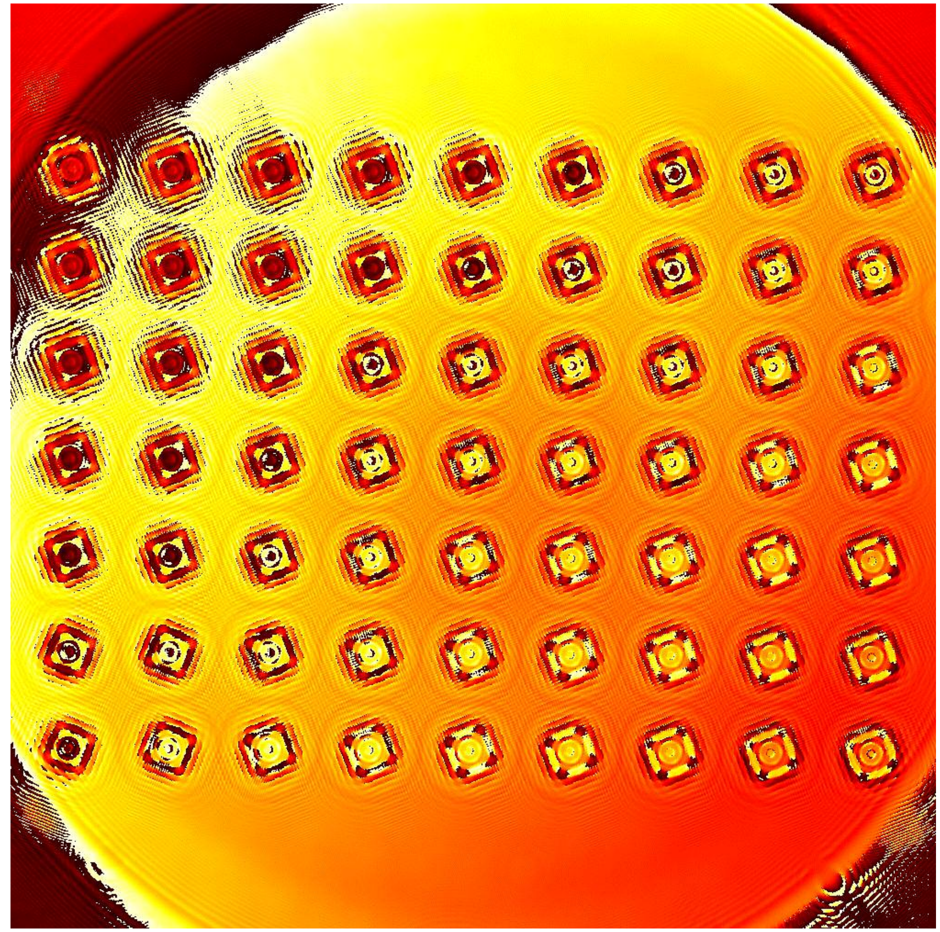
Intensity



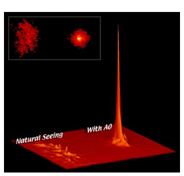
Calculated amplitude and phase of the field at the CCD



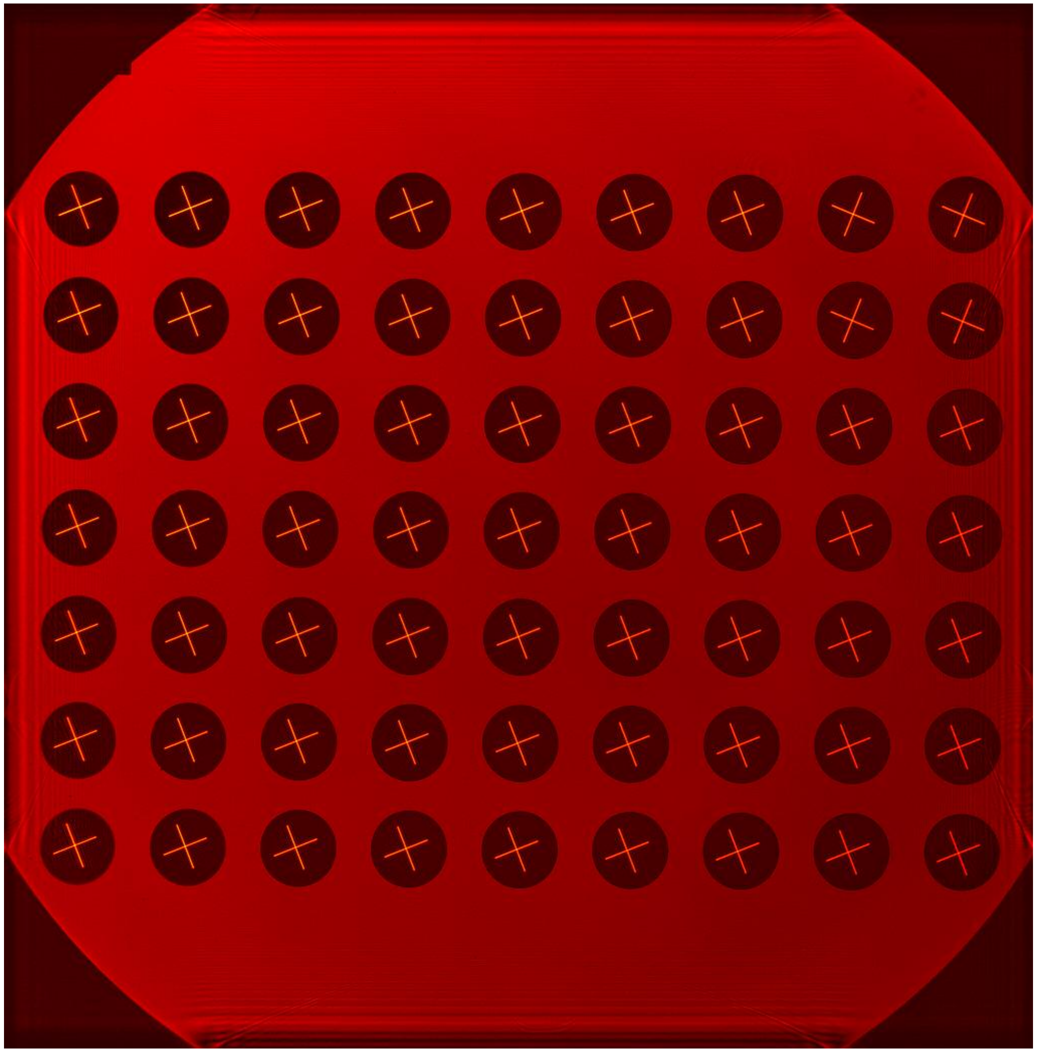
Amplitude



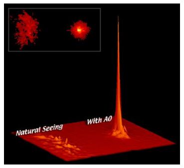
Phase



Back-propagation of the field to the DCG



Expanded view of a single fiducial



Measurement of the wrapped phase and the amplitude at the CCD using phase-shifting interferometry

Step 1: Measure a set of intensity frames with phase step $\alpha = 2\pi\nu_o\left(t_j - t_{j-1}\right)$

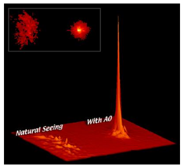
Here ν_0 is the desired slope of the phase ramp and the t_j are the equally spaced times at which the intensity frames are measured. (Actually, they are the centers of the integration periods). The intensity frames I_j are functions of the pixel position and have the form:

$$I_j = A_o + B_o \cos\left[2\pi\nu_o t_j + \theta\right]$$

where A_0, B_0, and θ are functions of the pixel position. It is the amplitude B_0 and the phase θ that we wish to measure.

Step 2: Compute the complex intensity $G = \sum_j w_j I_j \exp\left(-i\varphi_j\right)$

Here $\phi_j = 2\pi\nu t_j$ where ν is the actual slope of the phase ramp. The complex weights w_j have the time reversal symmetry $w_{P-j+1} = w_j^*$ where P is the total number of phase steps. The time reversal symmetry is present because all the constraints have time reversal symmetry.



Measurement of the wrapped phase and the amplitude at the CCD using phase-shifting interferometry (continued #1)

The complex intensity G has the form

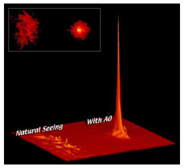
$$G = G_o + G_+(\nu_o)\exp(+i\theta) + G_-(\nu_o)\exp(-i\theta)$$

Here the coefficients of the exponential factors in the phase θ are given by

$$G_o = A_o \sum_j w_j \exp(-i\varphi_j)$$

$$G_+(\nu_o) = \frac{1}{2} B_o \sum_j w_j \exp(2\pi i \nu_o t_j - i\varphi_j)$$

$$G_-(\nu_o) = \frac{1}{2} B_o \sum_j w_j \exp(-2\pi i \nu_o t_j - i\varphi_j)$$



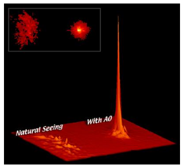
Measurement of the wrapped phase and the amplitude at the CCD using phase-shifting interferometry (continued #2)

When $\nu_0=\nu$, these equations simplify to:

$$G_+ = \frac{B_0}{2}\sum_j w_j$$

$$G_0 = A_0 \sum_j w_j \exp\left(-i\varphi_j\right)$$

$$G_- = \frac{B_0}{2}\sum_j w_j \exp\left(-2i\varphi_j\right)$$



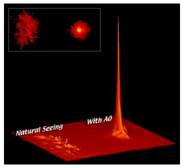
Measurement of the wrapped phase and the amplitude at the CCD using phase-shifting interferometry (continued #2)

When $\nu_0 = \nu$, these equations simplify to:

$$G_+ = \frac{B_0}{2} \sum_j w_j$$

$$G_0 = A_0 \sum_j w_j \exp\left(-i\varphi_j\right)$$

$$G_- = \frac{B_0}{2} \sum_j w_j \exp\left(-2i\varphi_j\right)$$



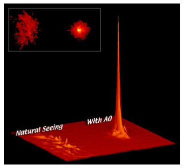
Measurement of the wrapped phase and the amplitude at the CCD using phase-shifting interferometry (continued #3)

In order for the phase of G to always be θ when $\nu_0=\nu$, we must have G_+ be positive real and both G_- and G_0 be zero. This gives the fundamental set of equations:

$$\sum_j w_j \text{ is positive real}$$

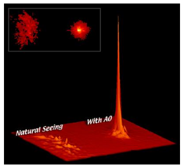
$$\sum_j w_j \exp\left(-i\varphi_j\right) = 0$$

$$\sum_j w_j \exp\left(-2i\varphi_j\right) = 0$$



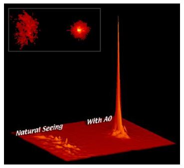
We use a twelve bucket $\pi/4$ phase step least square PSI algorithm

j	w_j
1	-1.116771628155 + 2.961574053933 i
2	+4.806376479712 + 2.586172699807 i
3	+1.822030515151 + 2.586172699807 i
4	+7.745178623018 + 2.961574053933 i
5	+6.628406994863
6	+6.628406994863
7	+6.628406994863
8	+6.628406994863
9	+7.745178623017 - 2.961574053933 i
10	+1.822030515151 - 2.586172699807 i
11	+4.806376479712 - 2.586172699807 i
12	-1.116771628155 - 2.961574053933 i



Properties of this twelve bucket, $\pi/4$ phase step PSI algorithm

- Insensitive to second order to errors in the slope of the phase ramp
- Insensitive to first order to a curvature of the phase ramp
- Insensitive to first order to a linear drift of the laser power
- Insensitive to CCD nonlinearities up to and including the sixth order provided that no other error sources are present
- Has a high noise figure of merit of 2.6096. This is more than 75% of the maximum possible noise figure of merit for any twelve bucket PSI algorithm.



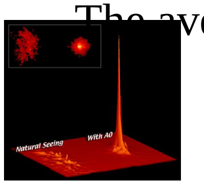
Delayed phase unwrapping

Typically we average the wrapped phase and amplitude data from at least eight sets of intensity frames together. The program keeps two-dimensional arrays of the running sums of the wrapped phases and of the amplitudes. The last wrapped phase array is remembered after all the pixels are put on the correct branches by adjusting the phases by the correct multiples of 2π radians. When new wrapped phase and amplitude arrays are measured, the phases of the pixels in the new wrapped phase array are put on the same branches as the last remembered adjusted wrapped phase array. This wrapped phase array then becomes the new last remembered adjusted wrapped phase array. This means that for each pixel that that multiple of 2π is added to the phase in radians so that that pixel will be within π radians of its last phase value after being put on the correct branch. This method will work only if the interferometer is sufficiently stable so that the phase nowhere changes by more than half a fringe between measurements of successive sets of twelve intensity frames. For each pixel (i,j), we thus have the sequence of phases in radians:

$$\phi_1(i,j) \quad \phi_2(i,j) \quad \phi_3(i,j) \quad \ldots \quad \phi_i(i,j) \quad \ldots \quad \phi_N(i,j)$$

such that $\left|\phi_k(i,j) - \phi_{k-1}(i,j)\right| \le \pi$ for k = 2, 3, …, N

The averaged phases are given by $$\phi_{averaged}(i,j) = \frac{1}{N}\sum_{k=1}^{N}\phi_k(i,j)$$

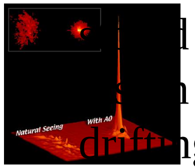


Complex phasor averaging

The problem with separately averaging the adjusted wrapped phases and the amplitudes is that the although the phase errors go to zero as $1/\sqrt{N}$, where N is the number of measurements averaged together, the amplitude errors do not. To see this, suppose that the amplitude really is zero at a given pixel. With noise, the measured amplitude will always be nonzero and positive and so can never average to zero. If one did not have to numerically propagate the wavefront, this would not be a significant concern.

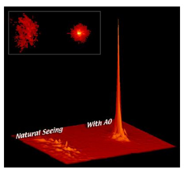
The problem with simply averaging the complex phasors is that the phase of each pixel may drift a fringe or two over the course of time of taking all the individual measurements that make up an average. Therefore the average of the complex phasors may be near zero. If the tilt fringes drift with time, the amplitude will be reduced by differing factors across the image. Furthermore, the signal to noise will everywhere be reduced.

The way around this is to select a region of the wrapped phase two-dimensional array where the data is always known to be good and for which it is known that the phase may be unwrapped so as to be smoothly connected. For any measurement, we first adjust the phases to be on the correct branches as before. Of course, if it is the first measurement, there is no adjustment. We then unwrap the phase in this region starting from a fixed pixel position. If it is the first measurement, the program simply stores the phases in this region. If it is a measurement after the first, the unwrapped phases in this region have subtracted from them the [illegible]l unwrapped phases in this region from the first measurement. This difference should be [illegible]n and tilts with possibly a small amount of focus. At present we do not correct for dri[illegible]ng focus.



Complex phasor averaging (continued)

We subtract from the piston and tilts from the entire two-dimensional phase array. We then compute the two-dimensional complex phasor array and sum this to the running sum complex phasor array. At the end, we divide by the total number of measurements N to obtain the average two-dimensional complex phasor array.



Huygen's integral transforms the wavefront from one surface to another

ABCD matrix

$$\begin{bmatrix} h' \\ n'u' \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} h \\ nu \end{bmatrix}$$

field

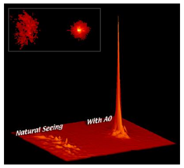
$$\tilde{u}_2(x_2) = \int \tilde{K}(x_2, x_1)\, \tilde{u}_1(x_1)\, dx_1$$

kernel

$$\tilde{K}(x_2, x_1) = \sqrt{\frac{j}{B\lambda_0}} \exp[-jk\,\rho(x_2, x_1)]$$

eikonal

$$\rho(x_2, x_1) \cong L_0 + \frac{1}{2B}\left(A x_1{}^2 - 2 x_1 x_2 + D x_2{}^2\right)$$



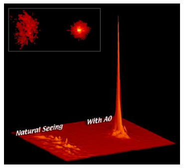
Properties of the Huygen's integral transformation

Conservation of power: $$\int_{-\infty}^{+\infty}\left[u_2(x)\right]^2 dx = \int_{-\infty}^{+\infty}\left[u_1(x)\right]^2 dx$$

Path independence and reversibility:

$$\int_{-\infty}^{+\infty} K^{(M_2 M_1)}(x_2, x_1)\, u_1(x_1)\, dx = \int_{-\infty}^{+\infty} K^{(M_2)}(x_2, x') \int_{-\infty}^{+\infty} K^{(M_1)}(x', x_1)\, u_1(x_1)\, dx_1\, dx'$$

where M_1 is the ABCD matrix from x_1 to x' and M_2 is the ABCD matrix from x' to x_2



The Huygen's integral must be evaluated by the 2D complex fast Fourier transform

The time to do a FFT on an NxN image is proportional to $N^2 \log_2 N^2$ compared to a time proportional to N^4 for direct integration. For a 1024x1024 image, the time required to directly do the integration would be about 50,000 times longer!

The value of dx_2 is uniquely determined by dx_1 and the ABCD matrix:

$$\frac{dx_1\, dx_2}{B\,\lambda_0} = \frac{1}{N}$$

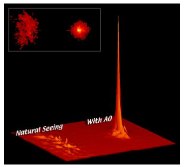
This is because the 1D discrete Fourier transform has the form:

$$H_n = \sum_{k=0}^{N-1} h_k\, e^{2\pi i k n / N}$$

If x_1=k dx_1 and x_2=n dx_2, then the x_1x_2 term in the exponential will have the form:

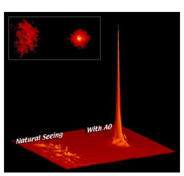
$$\exp\left(-2\pi i\, k\, n / N\right)$$

If the plane 2 image is not between $\pm N^*dx_2/2$, aliasing will occur.



The exponential factors $\exp\left(-j\frac{\pi}{B\lambda_0}A{x_1}^2\right) \quad \exp\left(-j\frac{\pi}{B\lambda_0}D{x_2}^2\right)$

represent quadratic phase factors that must be applied before and after the Huygen’s integral transformation, respectively.



The Huygen's integral transformation using [ABCD] matrices is used to transform between surfaces

$$\tilde{u}_2(x_2, y_2) = \iint \tilde{K}(x_2, x_1)\tilde{K}(y_2, y_1)u(x_1, y_1)dx_1\,dy_1$$

$$\tilde{K}(x_2, x_1) = \sqrt{\frac{j}{B\lambda_0}}\exp\left[-j\frac{\pi}{B\lambda_0}\left(Ax_1{}^2 - 2x_1x_2 + Dx_2{}^2\right)\right]$$

$$\tilde{K}(y_2, y_1) = \sqrt{\frac{j}{B\lambda_0}}\exp\left[-j\frac{\pi}{B\lambda_0}\left(Ay_1{}^2 - 2y_1y_2 + Dy_2{}^2\right)\right]$$

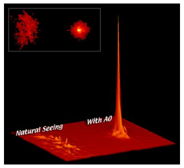
Two 2D fast FFTs are required for each Huygen's integral transformation:

1. Transform to the pinhole mirror plane from the first surface
2. Transform from the pinhole mirror plane to the second surface

Transforming to or from the pinhole mirror plane requires three steps:

1. Apply the quadratic phase factor due to the A $(x_1{}^2 + y_1{}^2)$ term
2. Do a fast 2D FFT
3. Apply the quadratic phase factor due to the D $(x_2{}^2 + y_2{}^2)$ term

Note that j = -i here.



Basic concept of the phase unwrapping

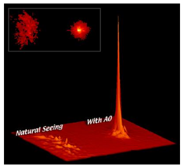
- Pixels are distingished as being either *wetable* or *unwetable*
  - Wetable pixels have good phase values. Unwetable pixels do not.

- Pick a wetable pixel and start pouring paint there. As it is flows from wetable painted pixels to wetable unpainted pixels, the phase is connected. The process stops when there are no more wetable unpainted pixels that adjoin the wetable painted pixels.

- If all the wetable pixels are connected, we are done. If not, pick a wetable unpainted pixel and start pouring a different colored paint.

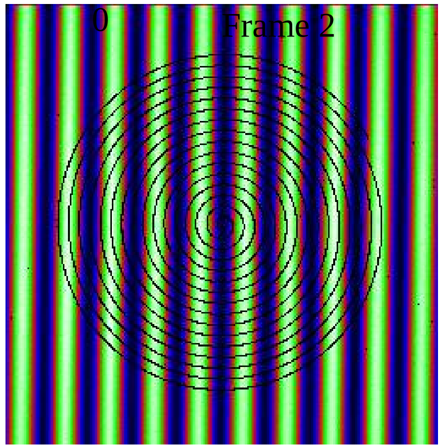
- If more than one color paint was used, choose the largest region of a single color.

- This algorithm demands that all the phase values be good. Therefore, before connecting the phase, it is required that all pixels having bad phase values be detected. Some possible tests are:
  - Any pixel having too low an amplitude is marked as bad.
  - Any 2x2 square of adjoining pixels for which the phases do not lie nearly on a plane are all marked as bad.
  - Any pixel for which the intensity values did not form a perfect enough sine wave of the right period is marked as bad.
  - Any pixel for which an intensity value was either too low or too high is marked as bad.

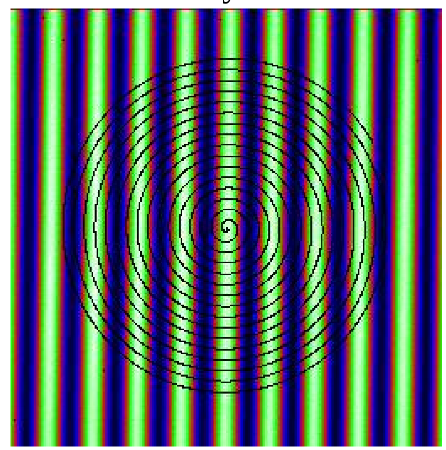


Phase unwrapping works for any connected topology

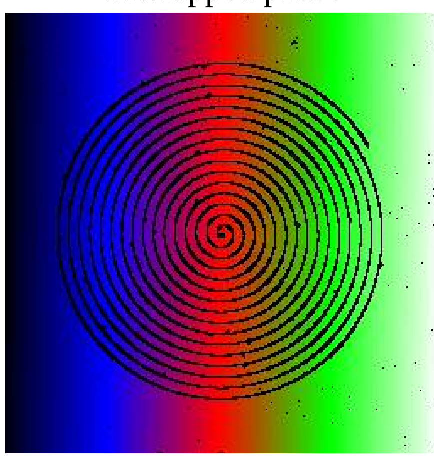
intensity frame 0



intensity frame 4



unwrapped phase



amplitude